

Circular loop operators in conformal field theories

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Abstract

We use the conformal group to study non-local operators in conformal field theories. A plane or a sphere (of any dimension) is mapped to itself by some subgroup of the conformal group, hence operators confined to that submanifold may be classified in representations of this subgroup. For local operators this gives the usual definition of conformal dimension and spin, but some conformal field theories contain interesting nonlocal operators, like Wilson or 't Hooft loops. We apply those ideas to Wilson loops in four-dimensional CFTs and show how they can be chosen to be in fixed representations of $SL(2, \mathbb{R}) \times SO(3)$.

Conformal field theories (CFTs) play an important role in physics. They arise naturally at the fixed points of the renormalization group flow and describe critical phenomena in a wide class of systems. From the theorists perspective CFTs offer an enlargement of the space-time symmetry group that puts constraints on the theory and makes it easier to study than a general quantum field theory.

Many field theories contain non-local observables, like Wilson loop operators, or topological defects, like Nielsen-Olesen vortices, or 't Hooft loops. Such objects may appear also in theories that have a conformal symmetry, for example Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In this note we propose some tools to study non-local operators in conformal field theories.

Let us recall the construction of local conformal operators. The conformal group in d -dimensional Euclidean space is $SO(d+1, 1)$. The subgroup that will keep a fixed point (the origin) invariant is $SO(d) \times \mathbb{R}$, comprising of rotations and the dilatation. Hence local operators may be classified by representations of this subgroup, the spin and conformal dimension.

To generalize this construction for non-local operators consider an n -dimensional sphere in \mathbb{R}^d . The subgroup of the conformal group that maps the sphere to itself is $SO(d-n) \times SO(n+1, 1)$. A simple way to see this symmetry is to map the sphere to a plane by a stereographic projection. $SO(n+1, 1)$ is the conformal symmetry in this plane and $SO(d-n)$ are the rotations around the plane. In the specific case of $n = 1$ and $d = 4$ on which we concentrate later we call these operators “circular loop operators” and the symmetry group is $SL(2, \mathbb{R}) \times SO(3)$ [1].

Our main proposition concerns any non-local operator localized on a sphere. The claim is

Operators localized on S^n in a CFT can be classified by representations of $SO(n+1, 1) \times SO(d-n)$ in much the same way that local operators are classified by spin and conformal dimension.

This statement follows immediately from the preceding discussion. In the remainder of the note we will develop some tools for analyzing loop operators in this setting and demonstrate it in a few examples.

Symmetry

An arbitrary CFT posses an $SO(d+1, 1)$ symmetry generated by translations P_μ , Lorentz transformations $M_{\mu\nu}$, the dilatation D and special conformal transformations

K_μ . Those may be realized on the fields by [2]

$$\begin{aligned}
[P_\mu, \Phi(x)] &= -i\partial_\mu\Phi(x), \\
[M_{\mu\nu}, \Phi(x)] &= -i(x_\mu\partial_\nu - x_\nu\partial_\mu + \Sigma_{\mu\nu})\Phi(x), \\
[D, \Phi(x)] &= -i(\Delta + x_\mu\partial_\mu)\Phi(x), \\
[K_\mu, \Phi(x)] &= -i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu - 2x_\mu\Delta + 2x^\nu\Sigma_{\nu\mu})\Phi(x).
\end{aligned} \tag{1}$$

$\Sigma_{\mu\nu}$ are Lorentz matrices acting on non-scalar fields.

Of those generators, the ones that leave the origin invariant are the Lorentz transformations and the dilatation which generate the subgroup $SO(d) \times \mathbb{R}$. Once we classify local operators by representations of this subgroup we can construct the full multiplet by considering the action of the other generators, P_μ and K_μ on the local operators. Since those carry mass dimension $+1$ and -1 they can be regarded as raising and lowering operators, and if a state is annihilated by all the K_μ it's called a highest-weight state, or a primary operator.

Instead we are interested in the subgroup that maps a sphere S^n of radius R given by $\sum_{i=1}^{n+1} (x^i)^2 = R^2$ to itself. This is generated by

$$J_i = \frac{R}{2}P_i + \frac{1}{2R}K_i \quad \text{and} \quad M_{ij} \quad \text{for } i, j = 1, \dots, n+1, \tag{2}$$

$$L_{i'} = \frac{R}{2}P_{i'} - \frac{1}{2R}K_{i'} \quad \text{and} \quad M_{i'j'} \quad \text{for } i', j' = n+2, \dots, d. \tag{3}$$

The first operators $\{J_i, M_{ij}\}$ generate $SO(n+1, 1)$ while the second set of operators are the generators of $SO(d-n)$. One can easily check using the representation (1) that the $SO(d-n)$ operators map every point on the sphere to itself, while when restricted to the sphere the M_{ij} and J_i only include the derivatives tangent to the surface of the sphere and hence map the sphere to itself.

A simple way to realize this symmetry is by writing \mathbb{R}^d in a special coordinate system. Starting with (η, Ω_n) as polar coordinate in \mathbb{R}^{n+1} and (ζ, Ω_{d-n-2}) in the remaining space, we define ρ and θ by

$$\sin \theta = \frac{\zeta}{\tilde{r}}, \quad \sinh \rho = \frac{\eta}{\tilde{r}}, \quad \tilde{r} = \frac{\sqrt{(\zeta^2 + \eta^2 - R^2)^2 + 4R^2\zeta^2}}{2R} = \frac{R}{\cosh \rho - \cos \theta}. \tag{4}$$

The flat space metric is then written as

$$\begin{aligned}
ds^2 &= d\eta^2 + \eta^2 d\Omega_n^2 + d\zeta^2 + \zeta^2 d\Omega_{d-n-2}^2 \\
&= \tilde{r}^2 (d\rho^2 + \sinh^2 \rho d\Omega_n^2 + d\theta^2 + \sin^2 \theta d\Omega_{d-n-2}^2).
\end{aligned} \tag{5}$$

One can immediately see that after dividing by the conformal factor \tilde{r}^2 one gets the metric on $\mathbb{H}_{n+1} \times S^{d-n-1}$, where \mathbb{H}_{n+1} is the $n+1$ -dimensional hyperbolic plane, also

known as Euclidean AdS_{n+1} . The sphere at $\eta = R$ and $\zeta = 0$ is mapped to the boundary of \mathbb{H}_{n+1} at $\rho \rightarrow \infty$, and is therefore invariant under the isometries of this space $SO(n+1, 1) \times SO(d-n)$.

Since the sphere is an orbit of the subgroup we may classify operators constrained on the sphere by representations of this subgroup. There are $(n+2)(d-n)$ generators of the full group that are not in this subgroup. Those transform in the $(\mathbf{n} + \mathbf{2}, \mathbf{d} - \mathbf{n})$ representation of $SO(n+1, 1) \times SO(d-n)$. To construct the operators explicitly we may start with the dilatation, which commute with all the M_{ij} . Acting on it with J_i and $L_{i'}$ will give the remaining operators in the coset.

The coset generators map operators on the sphere in different representations to each other in much the same way that P_μ and K_μ related local conformal operators of different dimensions.

Example: Wilson loops in four dimensions

Thus far we have considered operators on a sphere of arbitrary dimensions, In the rest of the paper we will concentrate on the case of $d = 4$ and $n = 1$, so the symmetry is $SL(2, \mathbb{R}) \times SO(3)$. For the $SL(2, \mathbb{R})$ generators we take $J_0 = -M_{12}$ and $J_\pm = J_1 \pm iJ_2$. For concreteness we label the angular coordinate in the plane of the circle by ψ and in the other two directions by ϕ .

The specific example we focus on is of Wilson loop operators in four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills. Another example one may consider is classical electro-magnetism. From the group theory analysis we know that they can be classified by representations of $SL(2, \mathbb{R}) \times SO(3)$.

Consider a Wilson loop along a circle of radius R

$$W = \text{Tr} \mathcal{P} e^{i \oint (A_\psi(R, \psi) + iR\Phi_6(R, \psi)) d\psi} . \quad (6)$$

This operator, with the inclusion of an extra adjoint scalar (Φ_6) in the exponent, is a very natural observable in the supersymmetric theory [3].

At the classical level both this operator and the one without the scalar term will be in the trivial representation of $SL(2, \mathbb{R})$, but this may be modified by quantum corrections. We expect the one with the scalar, which is supersymmetric, to remain in the trivial representation even after including quantum corrections.

If those were the only objects that can be studied by our classification it would be hardly worth the effort, but this is not the case. A general Wilson loop will not be circular, so it will not preserve the symmetry, but if the geometry is close enough to

the circle, we may expand it about the circular one. Small variations in the shape of the loop may be replaced by local insertions into the loop (see also [4])

$$\frac{\delta}{\delta x^\mu(s)} W \sim \text{Tr } \mathcal{P} i F_{\mu\nu} \dot{x}^\nu(s) e^{i \oint \dots} . \quad (7)$$

This operator is gauge invariant and circular, so may be classified in representations of $SL(2, \mathbb{R})$. In general, for any set of operators \mathcal{O}_i transforming in the adjoint representation of the gauge group we may consider the operator

$$W[\mathcal{O}_1(\psi_1) \cdots \mathcal{O}_k(\psi_k)] = \text{Tr } \mathcal{P} \left[\mathcal{O}_1(\psi_1) \cdots \mathcal{O}_k(\psi_k) e^{i \oint (A_\psi(R, \psi) + i R \Phi_6(R, \psi)) d\psi} \right] . \quad (8)$$

Those are the types of operators we suggest should be studied in this fashion.

Example: Smeared scalar operators

Let us now turn to calculating the dimension, and focus on the simple case of a single scalar insertion. Define the Fourier components

$$W[\mathcal{O}]^{(m)} = \frac{1}{2\pi} \text{Tr } \mathcal{P} \int d\psi' \mathcal{O}(\psi') e^{im\psi'} e^{i \oint (A_\psi(R, \psi) + i R \Phi_6(R, \psi)) d\psi} . \quad (9)$$

In perturbation theory the holonomy does not contribute at tree level, so in this case $W[\mathcal{O}]^{(m)}$ reduces to the smeared local operator

$$\mathcal{O}^{(m)} = \frac{1}{2\pi} \int d\psi \text{Tr } \mathcal{O}(\psi) e^{im\psi} . \quad (10)$$

As stated, in the ρ, ψ coordinate system, the $SL(2, \mathbb{R})$ symmetry is just the isometry of \mathbb{H}_2 . One can show that the action of the operators in (3) on scalar fields of dimension Δ is given by

$$\begin{aligned} [J_0, \text{Tr } \mathcal{O}] &= i \partial_\psi \text{Tr } \mathcal{O} , \\ [J_\pm, \text{Tr } \mathcal{O}] &= e^{\pm i\psi} \tilde{r}^{-\Delta} (-i \partial_\rho \pm \coth \rho \partial_\psi) \tilde{r}^\Delta \text{Tr } \mathcal{O} . \end{aligned} \quad (11)$$

When the operator is along the circle we may take $\rho \rightarrow \infty$ hence the action of J_\pm simplifies to

$$[J_\pm, \text{Tr } \mathcal{O}] = e^{\pm i\psi} (i\Delta \pm \partial_\psi) \text{Tr } \mathcal{O} . \quad (12)$$

Acting directly on the operators one finds after integration by parts

$$[J_0, \mathcal{O}^{(m)}] = m \mathcal{O}^{(m)} , \quad [J_\pm, \mathcal{O}^{(m)}] = -i(1 - \Delta \pm m) \mathcal{O}^{(m \pm 1)} . \quad (13)$$

The Casimir is $J^2 = (J_+ J_- + J_- J_+)/2 - J_0^2 = -\Delta(\Delta - 1)$, which is consistent with representations with principle quantum number¹ $j = \Delta$ or $j = 1 - \Delta$. Since the

¹On the representation of $SL(2, \mathbb{R})$ and notations, see [5].

representation includes all integer values of m it's in the continuous series and since for integer Δ the lowering operator J_- annihilates the operator with $m = 1 - \Delta$, it has $j = 1 - \Delta$. This representation is non-unitary and for integer Δ includes as sub-representations the states with $m > 1 - \Delta$, those with $m < \Delta - 1$ or their intersection, which is a finite-dimensional representation (the same as the unitary representations of $SU(2)$).

Instead of considering the Fourier modes of fields we can consider local insertions into the Wilson loop, say at $\psi = 0$ and $\psi = \pi$. Now we can look at the generators

$$\tilde{J}_0 = -iJ_1, \quad \tilde{J}_{\pm} = iJ_0 \mp iJ_2. \quad (14)$$

A local primary operator of dimension Δ at $\psi = 0$ will be annihilated by \tilde{J}_- and have an eigenvalue Δ for \tilde{J}_0 , acting with \tilde{J}_+ will give a tower

$$\text{Tr } \mathcal{O}(0), \quad -2 \text{Tr } \partial_\psi \mathcal{O}(0), \quad 4 \text{Tr } \partial_\psi^2 \mathcal{O}(0) - 2\Delta \text{Tr } \mathcal{O}(0), \quad \dots \quad (15)$$

with increasing values of \tilde{J}_0 that are in the discrete unitary representation \mathcal{D}_Δ^+ . In a similar way a primary local operator at $\psi = \pi$ will have $\tilde{J}_0 = -\Delta$ and will be the highest weight state in the conjugate representation \mathcal{D}_Δ^- . There are other operators that are in the continuous representations $\mathcal{C}_{1-\Delta}^\alpha$ in this basis with arbitrary α .

Note that it is possible to map flat \mathbb{R}^4 to $S^3 \times \mathbb{R}$ by a conformal transformation that will map $\psi = 0$ to past infinity and $\psi = \pi$ to future infinity. Then \tilde{J}_0 will correspond to time translation and the unitary representations will correspond to physical states in the Wick rotation of that space.

For local operators a simple way of calculating the dimension is through the two point function

$$\langle \text{Tr } \bar{\mathcal{O}}(x) \text{Tr } \mathcal{O}(0) \rangle \sim \frac{1}{x^{2\Delta}}. \quad (16)$$

One can study the representations of circular loop operators in a similar fashion. Consider the two point function of a loop operator of radius R and mode number m and another of radius η at ζ with an insertion at ψ

$$\langle W[\bar{\mathcal{O}}(\eta, \psi, \zeta)] W[\mathcal{O}]^{(m)} \rangle \quad (17)$$

As stated, at tree level we may replace the Wilson loop with the smeared local operator $\mathcal{O}^{(m)}$. If $\text{Tr } \mathcal{O}$ has dimension Δ , the two point function is given by

$$\langle W[\bar{\mathcal{O}}(\eta, \psi, \zeta)] W[\mathcal{O}]^{(m)} \rangle = \frac{1}{(2\pi)^{2\Delta+1}} \int d\psi' \frac{e^{im\psi'}}{(\zeta^2 + \eta^2 + R^2 - 2R\eta \cos(\psi - \psi'))^{2\Delta}}. \quad (18)$$

After writing this in terms of ρ and θ , this integral is

$$\frac{e^{im\psi}}{2\pi(8\pi^2 R\tilde{r})^\Delta} \int d\psi' \frac{e^{im\psi'}}{(\cosh \rho - \sinh \rho \cos \psi')^{2\Delta}} = \frac{e^{im\psi}}{(8\pi^2 R\tilde{r})^\Delta} \frac{\Gamma(\Delta + |m|)}{\Gamma(\Delta)} P_{-\Delta}^{-|m|}(\cosh \rho), \quad (19)$$

where $P_{-\Delta}^k$ is a version of the associated Legendre function which is defined on the positive real line but has a branch cut along $(-\infty, 1]$. It is equal to

$$P_{-\Delta}^k(\cosh \rho) = \left(\coth \frac{\rho}{2} \right)^k {}_2F_1 \left(\Delta, 1 - \Delta; 1 - k \mid -\sinh^2 \frac{\rho}{2} \right). \quad (20)$$

One can see that this function accompanied by the phase factor is an eigenfunction of the Laplacean of \mathbb{H}_2 with eigenvalue $-\Delta(\Delta - 1)$.

Acting with the generators J_i on the coordinates ρ and ψ one finds the relations

$$\begin{aligned} \langle (J_0 W[\bar{\mathcal{O}}(\eta, \psi, \zeta)]) W[\mathcal{O}]^{(-m)} \rangle &= m \langle W[\bar{\mathcal{O}}(\eta, \psi, \zeta)] W[\mathcal{O}]^{(-m)} \rangle \\ \langle (J_\pm W[\bar{\mathcal{O}}(\eta, \psi, \zeta)]) W[\mathcal{O}]^{-(m \pm 1)} \rangle &= -i(\Delta \pm m) \langle W[\bar{\mathcal{O}}(\eta, \psi, \zeta)] W[\mathcal{O}]^{(-m)} \rangle. \end{aligned} \quad (21)$$

Combining this result with the direct application of the symmetry generators on $W[\mathcal{O}]^{(m)}$ in (13) we can verify the Ward identities

$$J_i \langle W[\bar{\mathcal{O}}(\eta, \psi, \zeta)] W[\mathcal{O}]^{(m)} \rangle = 0, \quad (22)$$

up to contact terms.

The OPE

An important property of local operators in a CFT is the existence of the operator product expansion (OPE). It is possible to replace two nearby operators of dimensions Δ_1, Δ_2 with a series of operators of dimensions Δ_k

$$\mathcal{O}_2(x) \mathcal{O}_1(0) = \sum_k C_{21}^k x^{\Delta_k - \Delta_1 - \Delta_2} \mathcal{O}_k. \quad (23)$$

One may hope that a similar property applies to loop operators. To justify that, note that loop operators specify boundary conditions on \mathbb{H}_2 and if one considers two nearly coincident loops, they are both near the boundary. We may then look at the result in the bulk and write it in terms of one set of boundary conditions.

If our loop operators are made of smeared gauge invariant operators on the circles at $\rho = \infty$ and (ρ, θ) , the OPE is inherited from that of the local operators as

$$\mathcal{O}_2^{m_2}(\rho, \theta) \mathcal{O}_1^{m_1} = \sum_k C_{21}^k \mathcal{O}_k^{m_1 + m_2} (2R\tilde{r})^{\delta/2} \frac{\Gamma(-\delta/2 - m_2)}{\Gamma(-\delta/2)} P_{+\delta/2}^{-|m_2|}(\cosh \rho), \quad (24)$$

where $\delta = \Delta_k - \Delta_1 - \Delta_2$. In comparing to the OPE of local operators, the spatial dependence is more complicated and includes both the \tilde{r} dependence as well as the associated Legendre function.

Descendants

Similarly to local operators, also in the case of circular loop operators different $SL(2, \mathbb{R})$ representations may be related to each other by the action of generators in the coset $SO(5, 1)/SL(2, \mathbb{R}) \times SO(3)$. This coset is nine-dimensional with the operators transforming in the $(\mathbf{3}, \mathbf{3})$ of $SL(2, \mathbb{R}) \times SO(3)$. One element in the coset is the dilatation operator D . This operator commutes with both J_0 and $L_0 = -M_{34}$ and it's easy to write down the other 8 generators by commuting it with the raising and lowering operators.

Acting with the dilatation operator on the Wilson loop (6) gives

$$W = R \text{Tr} \mathcal{P} \int d\psi (F_{\eta\psi}(\psi) + i R D_\eta \Phi(\Psi)) e^{i \oint (A_\psi + i R \Phi) d\psi'}. \quad (25)$$

This operator is in the $j = -1$ representation of $SL(2, \mathbb{R})$ since the Wilson loop itself is in the trivial representation. Acting with J_\pm on this operator will insert phase factors $e^{\pm i\psi}$ into the integral. The action of the $SO(3)$ generators will replace η in the field-strength and derivative with the 3 and 4 directions.

In general the action of D on a circular operator in the representation with $j = 1 - \Delta$ will yield a reducible representation—the product of that representations with $j = -1$, including states with $j = -\Delta$, $1 - \Delta$ and $2 - \Delta$. One can check this explicitly for our example by repeating the calculation in (13) on the Fourier modes of $[D, \text{Tr} \mathcal{O}]$. We will call a circular loop operator a primary if the result will include only the representation with $j = -\Delta$. Indeed if we consider an insertion \mathcal{O} into the loop such that $\text{Tr} \mathcal{O}$ is a primary local operator, the resulting loop operator will be a primary (at tree level) by this definition.

Outlook

We have presented a method of classifying non-local operators in conformal field theories that are constrained to spherical or planar subspaces. This classification is based on the subgroup of the conformal group that preserves this subspace. We demonstrated this on the example of Wilson loops in four dimensions, which may be organized into representations of $SL(2, \mathbb{R}) \times SO(3)$.

The simplest circular Wilson loop in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory seems to be in the trivial representation. For other Wilson loops we suggested expanding nearly-circular operators in terms of circular loops with insertions into them. Those will fit into other representations of the symmetry group.

In the examples we studied we calculated the representations only at tree level and have so far not considered quantum corrections. The dimensions of local operators get

corrected in perturbation theory, and so should the $SL(2, \mathbb{R})$ representations. Those may be calculated by looking at the two-point function of Wilson loops, as was done for the simplest two circles in [6]. The result is not expected to remain Legendre functions with modified index, rather the two-point function would be a representation of the symmetry whose generators are modified by quantum corrections.

One may wish to study other objects, for example 't Hooft loops. Those may be described semiclassically by a magnetic flux in some $U(1)$ subgroup of the gauge group sourced along the circle. After the map to $\mathbb{H}_2 \times S^2$ those would correspond to a constant magnetic flux on S^2 [1]. Since this has no structure on \mathbb{H}_2 it would seem natural to conjecture that this too is in the trivial representation of $SL(2, \mathbb{R})$.

Again, there should be generalizations corresponding to small deformations of the circular 't Hooft loop. It may be possible to study those by a similar semiclassical description, where now other components of the electromagnetic field would be excited. We leave the study of these objects to the future.

In the case of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory the basic circular loop preserves half the supersymmetries and has some remarkable properties when calculated both in perturbation theory and in the dual string theory on $AdS_5 \times S^5$ [7]. One consequence of the supersymmetry is that the group presented above is enlarged to a supergroup with 16 fermionic generators. This leads to a much richer structure that will be studied elsewhere. Furthermore, in that case one can look at the string theory duals of those operators described by classical string solutions in $AdS_5 \times S^5$ and their $SL(2, \mathbb{R})$ representations should be calculable there too.

We have applied those ideas only to four-dimensional theories but the same could be done in arbitrary dimensions. In particular it would be interesting to study the spherical surface observable in the six-dimensional theory dual to $AdS_7 \times S^4$.

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